Chromatic Symmetric Function of Cycle Chains

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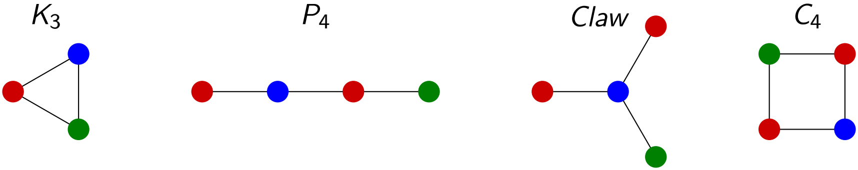
Graph Coloring

We want to color a graph *properly*—such that no two adjacent vertices have the same color. How many ways can we do this with at most $x = 5$ colors?



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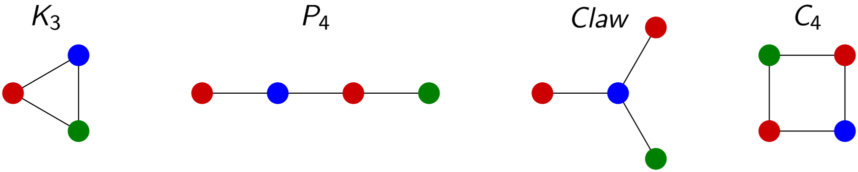

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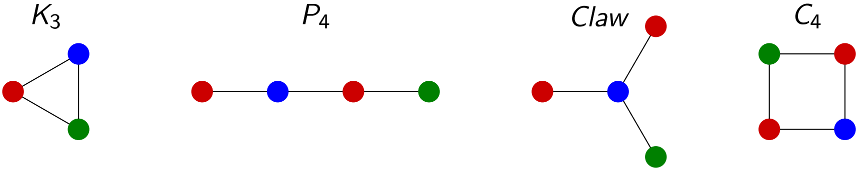

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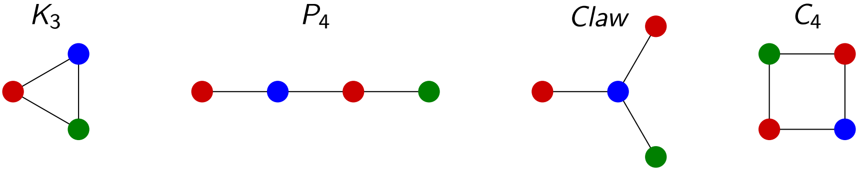

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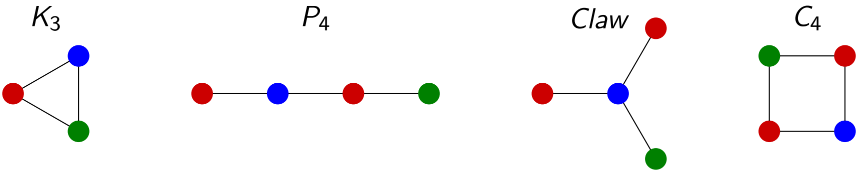

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For C_4, we must resort to casework on whether vertex diagonal vertices have the same color. We get $\boxed{260 \text{ ways}}$.

Chromatic Polynomial

Birkhoff defined the chromatic polynomial trying to solve the 4-color theorem.

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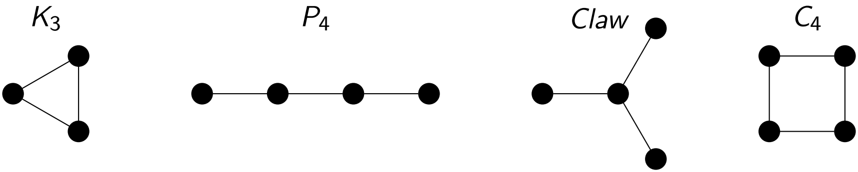

Example

The chromatic polynomials of the graphs above are

$$\chi_{K_3}(x) = x(x-1)(x-2), \qquad \chi_{P_4}(x) = x(x-1)^3,$$
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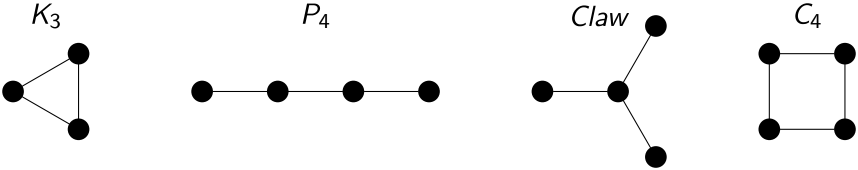

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Note that the chromatic polynomial of the claw and P_4 are the same.

A Symmetric Generalization

Stanley defined a generalization of this polynomial in [10].

Definition (Proper Coloring)

A *proper coloring* of graph G is a function $\kappa : V(G) \to \mathbb{N}$ such that if $(i, j) \in E(G)$, then $\kappa(i) \neq \kappa(j)$.

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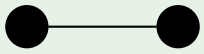

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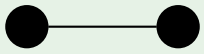

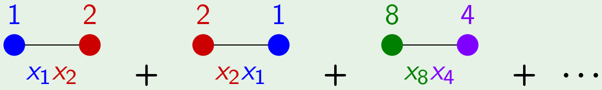

Say I color the left vertex "1" and the right vertex "2". Then, I can associate with that proper coloring the monomial x_1x_2. For every proper coloring, I can associate a monomial of degree 2.

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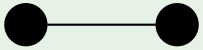

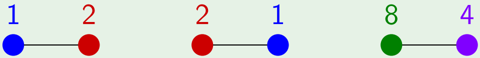

$$x_1x_2 \quad + \quad x_2x_1 \quad + \quad x_8x_4 \quad + \quad \cdots$$

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In general, we get that

$$X_{P_2}(\boldsymbol{x}) = (x_1x_2 + x_1x_3 + \cdots) + (x_2x_1 + x_2x_3 + \cdots) + \cdots = \sum_{\substack{i,j \in \mathbb{N} \\ i<j}} 2x_ix_j.$$

Chromatic Symmetric Function Reasoning

One cool property of the chromatic symmetric function is that it seems to be able to differentiate trees.

Conjecture

Non-isomorphic trees have different chromatic symmetric functions.

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Chromatic Symmetric Function to Chromatic Polynomial

If we know the $X_G(\boldsymbol{x})$, we can find $\chi_G(x)$ for any x, with

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The chromatic symmetric function can be used to find acyclic orientations of a graph, and is closely related to Jacobi Trudi matrices, Hessenberg varieties, and the characters of Kazhdan–Lusztig elements of the Hecke algebra [1, 2, 11].

Chromatic Symmetric Function Problems

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The chromatic symmetric function for P_4 can be written as

$$X_{P_4}(\boldsymbol{x}) = \sum_{\substack{i,j \in \mathbb{N} \\ i<j}} 2x_i^2 x_j^2 + \sum_{\substack{i,j,k \in \mathbb{N} \\ j<k \\ i \neq j, i \neq k}} 4x_i^2 x_j x_k + \sum_{\substack{i,j,k,l \in \mathbb{N} \\ i<j<k<l}} 24 x_i x_j x_k x_l.$$

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It turns out we can write the chromatic symmetric function as the finite sum of basis functions. We will work with the e-basis.

Elementary Basis

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We define $e_{(a_1,a_2,\ldots,a_n)} = e_{(a_1)} \cdots e_{(a_n)}$.
For example, $e_{(4,3,1,1)} = e_{(4)} \cdot e_{(3)} \cdot e_{(1)} \cdot e_{(1)}$.

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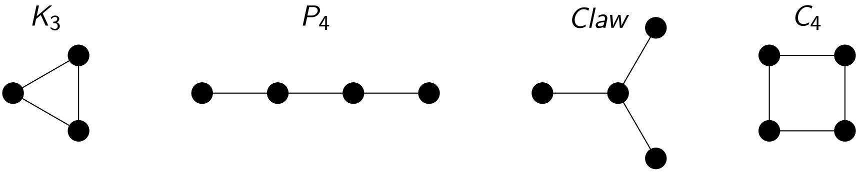

Example

The chromatic symmetric functions for the graphs above are

$$X_{K_n}(\boldsymbol{x}) = n!e_{(n)}, \qquad X_{P_4}(\boldsymbol{x}) = 2e_{(2,2)} + 2e_{(3,1)} + 4e_{(4)},$$
$$X_{C_4}(\boldsymbol{x}) = 2e_{(2,2)} + 12e_{(4)}, \quad X_{\text{claw}}(\boldsymbol{x}) = e_{(2,1,1)} - 2e_{(2,2)} + 5e_{(3,1)} + 4e_{(4)}.$$

Positivity Conjectures

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Conjecture (Simplified Stanley-Stembridge Conjecture [9, 11])

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Some graphs, including paths, cycles, cliques connected at single vertices, and a few other families of graphs have all been proven to be e-positive [3, 4, 5, 6, 7, 12, 13].

Our Positivity Theorem

Theorem (Tom, V.)

Adjacent cycle chains, which are graphs formed by connecting cycles at adjacent vertices, are e-positive.
Adjacent cycle+clique chains are also e-positive.

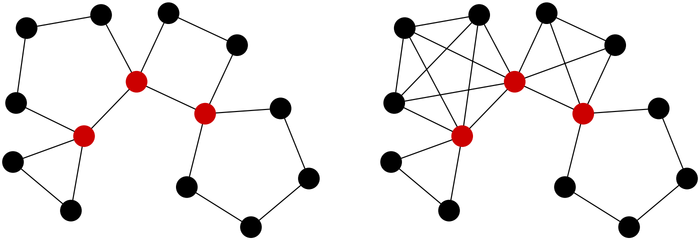
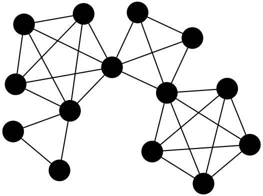


Figure: Adjacent cycle chain $C_3 + C_5 + C_4 + C_5$ next to cycle+clique chain $C_3 + K_5 + K_4 + C_5$.

Motivation

My mentor proved cliques connected at single vertices are e-positive.

Figure: Clique chain $K_3 + K_5 + K_4 + K_5$.



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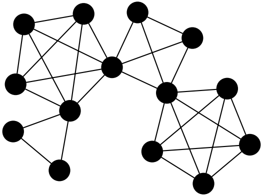
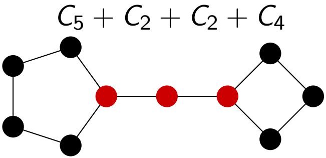
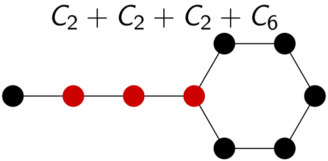


The *e*-positivity of cycles and graphs related to cycles (like cycle chords, dumbbells, and tadpoles) has been studied, but not cycle chains.

Figure: Dumbbell and Tadpole graphs.





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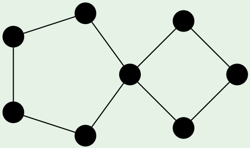
The chromatic symmetric function of the graph below can be calculated as

$$X_{C_5+C_4}(\boldsymbol{x}) = 6e_{(3,2,2,1)} + 6e_{(3,3,2)} + 20e_{(4,2,2)} + 18e_{(4,3,1)} + 40e_{(5,2,1)} + 54e_{(5,3)} + 108e_{(6,2)} + 72e_{(7,1)} + 96e_{(8)},$$

which indeed has positive coefficients.



Example

The formula works for large graphs!

Example

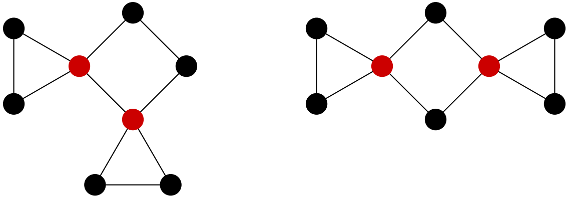
For $G = C_3 + C_5 + C_4 + C_5$, the chromatic symmetric function is

$$X_G(\boldsymbol{x}) = 24e_{(3,3,2,2,2,2)} + 56e_{(3,3,3,2,2,1)} + 48e_{(3,3,3,3,2)} + 116e_{(4,3,2,2,2,1)} + 32e_{(4,3,3,2,1,1)} + 348e_{(4,3,3,2,2)} + 160e_{(4,3,3,3,1)} + 12e_{(4,4,2,2,1,1)} + 80e_{(4,4,2,2,2)} + 112e_{(4,4,3,2,1)} + 32e_{(4,4,3,3)} + 24e_{(5,2,2,2,2,1)} + 88e_{(5,3,2,2,1,1)} + 544e_{(5,3,2,2,2)} + 648e_{(5,3,3,2,1)} + 480e_{(5,3,3,3)} + 496e_{(5,4,2,2,1)} + 24e_{(5,4,3,1,1)} + 868e_{(5,4,3,2)} + 96e_{(5,5,2,1,1)} + 1112e_{(5,5,2,2)} + 248e_{(5,5,3,1)} + 64e_{(5,5,4)} + 72e_{(6,2,2,2,1,1)} + 128e_{(6,2,2,2,2)} + 40e_{(6,3,2,1,1,1)} + 1176e_{(6,3,2,2,1)} + 152e_{(6,3,3,1,1)} + 1488e_{(6,3,3,2)} + 180e_{(6,4,2,1,1)} + 832e_{(6,4,2,2)} + 360e_{(6,4,3,1)} + 2288e_{(6,5,2,1)} + 792e_{(6,5,3)} + 360e_{(6,6,1,1)} + 1008e_{(6,6,2)} + 48e_{(7,2,2,1,1,1)} + 668e_{(7,2,2,2,1)} + 864e_{(7,3,2,1,1)} + 1996e_{(7,3,2,2)} + 1232e_{(7,3,3,1)} + 1136e_{(7,4,2,1)} + 512e_{(7,4,3)} + 480e_{(7,5,1,1)} + 3012e_{(7,5,2)} + 2008e_{(7,6,1)} + 896e_{(7,7)} + 892e_{(8,2,2,1,1)} + 1120e_{(8,2,2,2)} + 168e_{(8,3,1,1,1)} + 3332e_{(8,3,2,1)} + 1440e_{(8,3,3)} + 192e_{(8,4,1,1)} + 1344e_{(8,4,2)} + 1536e_{(8,5,1)} + 1344e_{(8,6)} + 320e_{(9,2,1,1,1)} + 3336e_{(9,2,2,1)} + 1304e_{(9,3,1,1)} + 3216e_{(9,3,2)} + 680e_{(9,4,1)} + 1176e_{(9,5)} + 2620e_{(10,2,1,1)} + 3104e_{(10,2,2)} + 2816e_{(10,3,1)} + 576e_{(10,4)} + 480e_{(11,1,1,1)} + 5548e_{(11,2,1)} + 1824e_{(11,3)} + 2072e_{(12,1,1)} + 3456e_{(12,2)} + 2920e_{(13,1)} + 1344e_{(14)}.$$

Other Cycle Chains

What about non-adjacent cycle chains?

Figure: Adjacent cycle chain $C_3 + C_4 + C_3$ on the left, next to a non-adjacent cycle chain.



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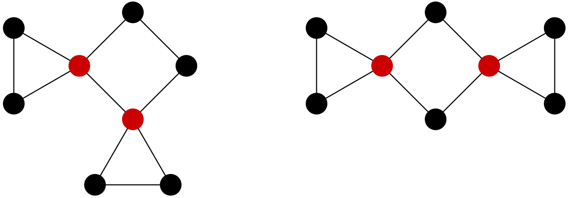

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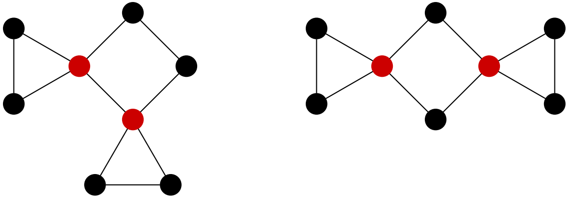

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$$X_G(\boldsymbol{x}) = 16e_{(3,3,1,1)} - 24e_{(3,3,2)} + 72e_{(4,3,1)} + 80e_{(4,4)} + 24e_{(5,2,1)} + 24e_{(5,3)} + 40e_{(6,1,1)} + 48e_{(6,2)} + 128e_{(7,1)} + 96e_{(8)}.$$

Acknowledgements

I would like to thank my mentor, Dr. Foster Tom, for introducing me to the topic of chromatic symmetric functions, helping me draft this paper, verify my equations, and simplify my proofs. This paper could not have been done without him.

I would also like to thank the PRIMES-USA program for introducing math research to me, pairing me up with my mentor, and providing insightful comments on my writing.

Lastly, I want to thank my parents for supporting me during this journey, from helping me keep my schedule on track to listening to my presentation.

References I

Alex Abreu and Antonio Nigro.
Parabolic lusztig varieties and chromatic symmetric functions, 2022, 2212.13497.

Alex Abreu and Antonio Nigro.
Splitting the cohomology of hessenberg varieties and e-positivity of chromatic symmetric functions, 2023, 2304.10644.

Per Alexandersson and Greta Panova.
Llt polynomials, chromatic quasisymmetric functions and graphs with cycles.
Discrete Mathematics, 341(12):3453–3482, December 2018.

Farid Aliniaeifard, Victor Wang, and Stephanie van Willigenburg.
The chromatic symmetric function of a graph centred at a vertex, 2021, 2108.04850.

References II

Soojin Cho and Jaehyun Hong.
Positivity of chromatic symmetric functions associated with Hessenberg functions of bounce number 3.
Electron. J. Combin., 29(2):Paper No. 2.19, 37, 2022.

Soojin Cho and JiSun Huh.
On e-positivity and e-unimodality of chromatic quasi-symmetric functions.
SIAM J. Discret. Math., 33:2286–2315, 2017.

Samantha Dahlberg.
Triangular ladders $p_{d,2}$ are e-positive, 2019, 1811.04885.

Samantha Dahlberg, Adrian She, and Stephanie van Willigenburg.
Schur and e-positivity of trees and cut vertices, 2019, 1901.02468.

References III

Mathieu Guay-Paquet.
A modular relation for the chromatic symmetric functions of (3+1)-free posets, 2013, 1306.2400.

Richard P Stanley.
A symmetric function generalization of the chromatic polynomial of a graph.
Advances in Mathematics, 111(1):166–194, 1995.

Richard P Stanley and John R Stembridge.
On immanants of jacobi-trudi matrices and permutations with restricted position.
Journal of Combinatorial Theory, Series A, 62(2):261–279, 1993.

Foster Tom.
A signed e-expansion of the chromatic quasisymmetric function, 2024, 2311.08020.

References IV

David G. L. Wang.
All cycle-chords are e-positive, 2024, 2405.01166.